Math 259A Lecture 17 Notes

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1 Group von Neumann Algebras for ICC Groups

1.1 ICC group von Neumann algebras

Last time, we introduced Ł(Γ), the group von Neumann algebra of Γ. This is the weak operator closure of $\operatorname{span} \lambda(\Gamma)$, where λ is the left regular representation. We saw that $L(\Gamma)$ and $R(\Gamma)$ has the faithful, SO-continuous (and hence WO-continuous) trace state $\tau(x) := \langle x\xi_e, \xi_e \rangle$. This implied that $L(\Gamma)$ is a finite von Neumann algebra.

Theorem 1.1. *$L(\Gamma)$ is a factor if and only if Γ is infinite conjugacy class (i.e. for any $g \neq e$, $\{hgh^{-1} : h \in \Gamma\}$ is infinite).*

Proof. ($\implies$): We did this last time.

($\impliedby$): If Γ is ICC but $z \in Z(\Gamma) \setminus \mathbb{C}1$, then $z(\xi_e) = \sum c_g \xi_g \in \ell^2(\Gamma)$ with $c_{g_0} \neq 0$ for some $g_0 \neq e$. Then for any $h, g \in \Gamma$,

$$\langle \lambda(g) z \lambda(g)^*(\xi_e), \xi_h \rangle = \langle z\xi_e, \xi_h \rangle = c_h.$$

On the other hand, $\lambda(g)^*(\xi_e) = \rho(g)(\xi_e)$, which commutes with z and $\lambda(g)$, so

$$\langle \lambda(g) z \lambda(g)^*(\xi_e), \xi_h \rangle = \langle \rho(g)\lambda(g) z \xi_e, \xi_h \rangle = \left\langle z\xi_e, \lambda_{g^{-1}} \xi_{hg} \right\rangle = \left\langle \sum_g c_g \xi_g, \xi_{g^{-1}hg} \right\rangle = c_{g^{-1}hg}.$$

Take $h = g_0$. Then $c_{g_0} = c_{gg_0g^{-1}}$ for all g, which gives infinitely many equal nonzero coefficients. $\square$

Corollary 1.1. *If Γ is ICC, $L(\Gamma)$ is a II_1 factor.*

Example 1.1. Let S_∞ be the group of finite permutations of $\mathbb{N}$. Then S_∞ is ICC. It is also **locally finite**: for any finite $F \subseteq S_\infty$, there is a finite subgroup of S_∞ containing F.

Example 1.2. Let $\mathbb{F}_n$ be the free group on n generators. This is ICC.

Definition 1.1. Given two groups H_0, Γ_0, $\Gamma_0 \circlearrowright H^{\Gamma_0}$ by left multiplication on the coordinates: $g_0(h_g)_{g\in\Gamma_0} = (h_{g_0^{-1}g})_{g\in\Gamma_0}$. The **wreath product** is the semidirect product $H^{\Gamma_0} \rtimes \Gamma_0$.

Example 1.3. When $H = \mathbb{Z}/2\mathbb{Z}$ and Γ_0, the wreath product is called the **lamp lighter group**. You can think of this as an infinite row of lamps, each lit or unlit. This group is ICC.

Example 1.4. More generally, if $H_0, \Gamma_0 \neq \{1\}$ and Γ is infinite, the wreath product is ICC.

1.2 Distinguishing groups by their von Neumann algebras

More detailed description of $L(\Gamma)$.

Different groups can give rise to different group von Neumann algebras.

Theorem 1.2 (M-vN, 1943)**.** $L(S_\infty) \neq L(\mathbb{F}_2)$.

However, there is some collapsing that goes on.

Theorem 1.3. *All ICC locally finite Γ give the same $L(\Gamma)$.*

Here is an open question:
Are $L(\mathbb{F}_n)$ isomorphic or not for different n?

1.3 Multiplication operators on $\ell^2(\Gamma)$

Proposition 1.1. *Any $\xi \in \ell^2(\Gamma)$ defines operators $L_\xi, R_\xi : \ell^2(\Gamma) \to \ell^\infty(\Gamma)$ by*

$$L_\xi(\eta) = \xi\eta = \sum_{g,h} c_g b_h \xi_{gh}, \qquad \textit{where} \quad \xi = \sum_g c_g \xi_g, \eta = \sum_h b_h \xi_h.$$

Moreover, $\|L_\xi\|_{\mathcal{B}(\ell^2,\ell^\infty)} \leq \|\xi\|_{\ell^2}$.

Proof. This follows by Cauchy-Schwarz:

$$\sup_{g\in\Gamma} |\xi\eta(g)| = \sup_{g\in\Gamma} \left| \sum_{h\in\Gamma} c_h b_{h^{-1}g} \right| \leq \|\xi\|_{\ell^2} \|\eta\|_{\ell^2}.$$

□

Proposition 1.2. *$D(L_\xi) = L_\xi^{-1}(\ell^2) \subseteq \ell^2$ is a vector subspace, closed in L^2. Moreoverl L_ξ on $D(L_\xi)$ is a densely defined, closed operator on $\ell^2(\Gamma)$.*

Proof. We need to show that L_ξ has closed graph. That is, we need to show that if $\eta_i \to 0$ and $L_\xi(\eta_i) \to \eta$ in ℓ^2, then $\eta = 0$. □

We will do this next time.